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A microstructural approach to the mechanical response of composite systems with randomly oriented, short fibres

Part 1 *Theoretical analysis*

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This paper is concerned with a new deformation theory of composite systems with randomly oriented short fibres. The material system is regarded as a two-dimensional elastic matrix that contains a random arrangement of short, viscoelastic fibres. Due to the inherent randomness of the physical and geometrical characteristics of the microstructure, probabilistic concepts are used. In this regard, the significant field quantities involved in the deformation process are treated in the analysis as stochastic variables and the deformation process itself is seen as a stochastic process. The analysis is presented in a general form and, hence, is applicable to a large class of fibre–composite systems.

1. Introduction

This paper aims at formulating the deformation process in fibre–composite materials with time memory and having discrete microstructure. Fig. 1 shows a schematic illustration of a model of a two-dimensional composite system dealt with in the present analysis. The material system is seen to be an elastic matrix which contains a layer of randomly oriented, short, viscoelastic fibres. Examples of such materials are epoxy preimpregnated (PREPREG) composites reinforced with chopped fibres of cellulosic origin such as paper, nylon, polymeric fibres and related types. Information regarding the actual structure of such materials, their fabrication and applications is given by Lubin [1] and Richardson [2].

Traditionally, models that are based on continuum theory have been used for the prediction of the response behaviour of fibre–composites. In recent years, attempts have been made to modify the classical continuum approach by allowing for microscopic or "local" quantities to enter into the analysis, but without removing the main restrictions imposed by continuum physics on such formulations. In this context, "deterministic" models were suggested by Hill [3] Bürgel *et al.* [4] and Chou and Chou [5], among others.

Discrete composite systems, such as those dealt with in the present analysis, have, however, distinct features. Firstly, there exists a multitude of singular surfaces (such as internal interfaces between the fibres and the matrix) within a given domain of the macroscopic material body. Secondly, the fibrous elements of the structure of the system have a finite size and exhibit random physical and configurational properties that cannot readily be brought into line with the conventional deterministic macroscopic relations. In view of these facts, it is evident that a new approach based on the characteristics of the real microstructure should be followed. Thus, the response behaviour of the fibre–composite system is studied in this paper by using a probabilistic microstructural approach [6, 7]. In this approach, the mechanics of the discrete microstructure introduce the relevant field quantities as random variables or functions of such variables and their corresponding distribution functions. Further, the deformation behaviour of individual fibres, and their interactions with the matrix, are considered in this analysis to be time-dependent in nature. Hence, it appears appropriate in dealing with such systems to consider the deformation process itself as a stochastic process.

In order to describe the mechanical response of a composite system with the inclusion of the microstructure, it is necessary to consider the response of an actual structural domain of the material which, on a local scale, may differ considerably from an average response if the phenomenological approach were taken. Such local deviations in mechanical response which would be neglected by ignoring the microstructure are, on the other hand, directly related to basic properties of the non-homogeneous composite system.

In order to extend the analysis to the practical case of a two-dimensional composite, it is necessary to make use of "mesoscopic quantities" arising from considerations of the existence of a statistical ensemble of local, structural domains within an intermediate domain of the material specimen. Further, it is equally important to find a connection between the microscopic and the macroscopic response formulations. Thus, the analysis aims at the formulation of a set of "governing response equations" for the structured composite system which, in contrast to the conventional formulations, are based on the concepts of

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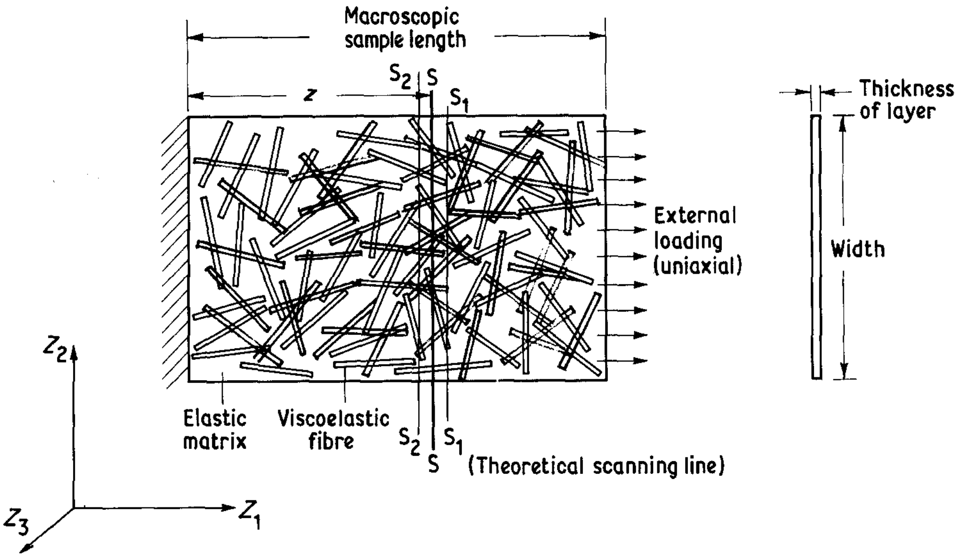

Figure 1 Two-dimensional model of a fibre-composite.

statistical theory and probabilistic micromechanics [6]. In this context, it has been found useful to employ operational representation of the various relations. Hence, the notion of a "material operator" characteristic of the mechanical response of an intermediate domain of the material is introduced. This material operator provides the connection between the stress field and occurring deformations within the intermediate domain under consideration. It contains in its argument those stochastic variables, or functions of such variables, distinctive of the microstructure within the intermediate domain.

2. Probabilistic, micromechanical response

2.1. A structural domain

A structural domain (α) is defined as the smallest region of the medium that represents the mechanical and physical characteristics of the microstructure at the "micro" level. This local domain, as shown in Fig. 2, is chosen to represent the characteristics of individual fibres together with those of the surrounding matrix material within a scanning area of dimensions $\mu \times \mu$.

Throughout the analysis, a superscript α to the left of the symbol refers to a structural domain of the composite system. The quantities referring to an individual fibre are denoted by a superscript f, while those referring to the adjacent matrix material are designated by a superscript m. The quantities referring to the interfacial bonding between the fibre and the matrix are identified by a superscript B.

For the description of the deformation kinematics of the microstructure within (α), it is convenient to use two local Cartesian frames of reference, i.e. ${}^{f}Y_i$ ($i = 1, 2, 3$) attached to the end of the fibre and ${}^{m}Y_k$

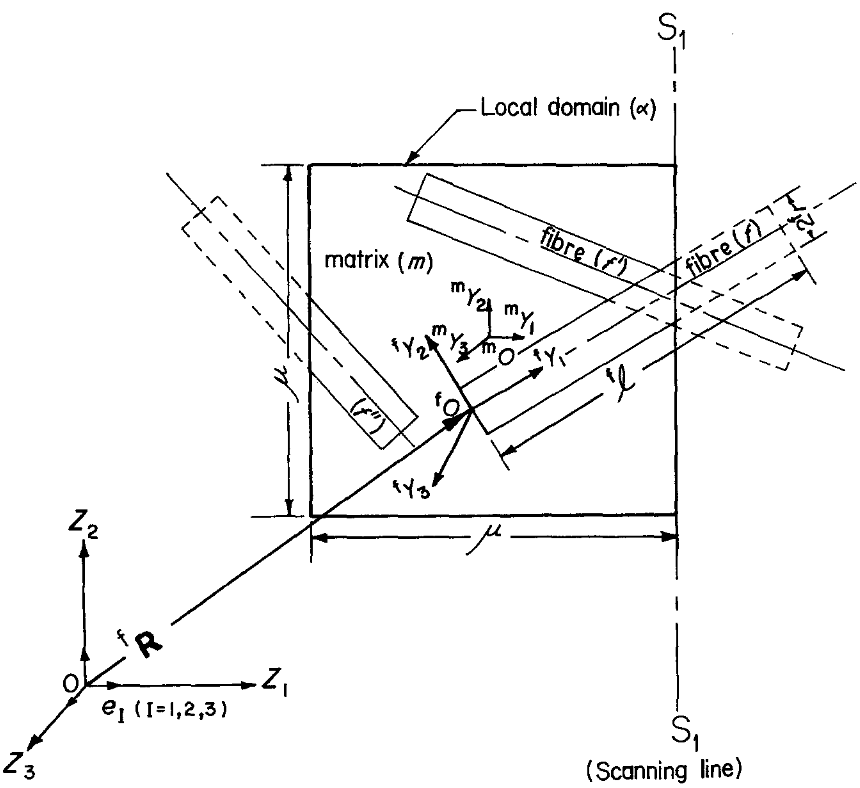

Figure 2 Model of a structural domain α.

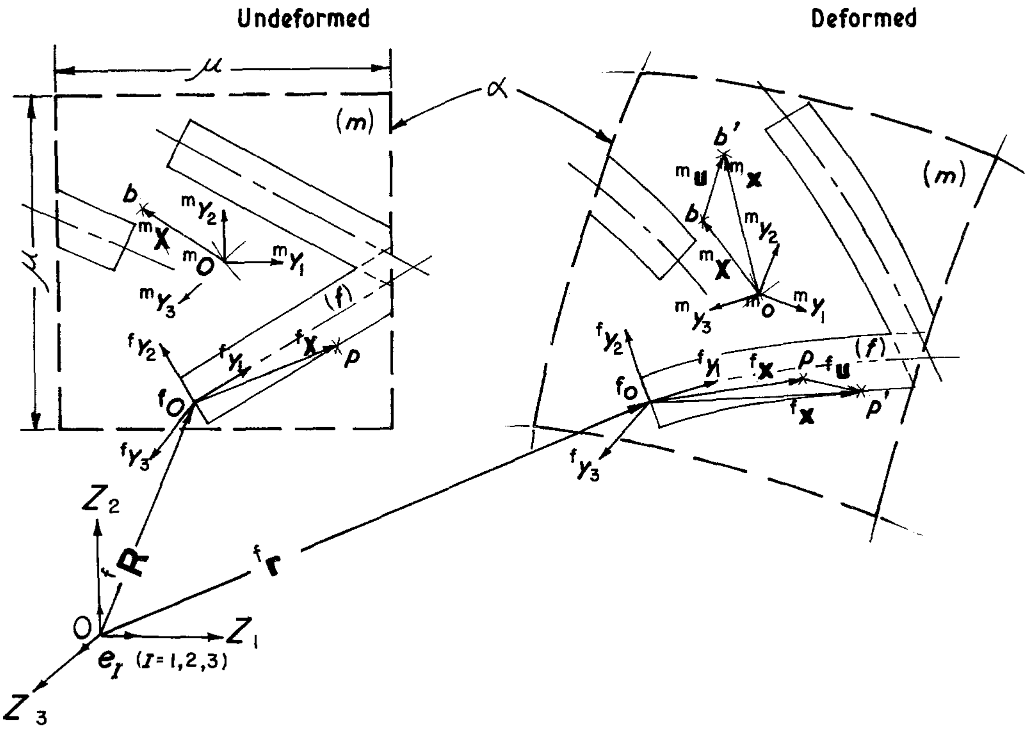

Figure 3 Deformation kinematics of a structural domain α.

(k = 1, 2, 3) attached locally to the adjacent matrix, Fig. 2. These coordinate-frames are to express the motion of the microstructure relative to an external fixed Cartesian frame Z_I (I = 1, 2, 3). In general, all kinematic parameters related to the underformed configuration of the microstructure are denoted in upper case, while those for the deformed configuration, are designated in lower case.

With reference to Fig. 3, let the position vector of a point within fibre f be denoted by ${}^{f}\boldsymbol{X}$, while the corresponding position vector of a point within the adjacent matrix, m be denoted by ${}^{m}\boldsymbol{X}$. The microdeformation in the fibre can then be expressed as

$$ {}^{f}u_i({}^{f}X_i, t) = {}^{f}x_i(t) - {}^{f}X_i \tag{1a}$$

and that in the matrix will be

$$ {}^{m}u_k({}^{m}X_k, t) = {}^{m}x_k(t) - {}^{m}X_k. \tag{1b}$$

The corresponding strain tensor may be written as

$$ \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1c}$$

where a comma indicates a partial differentiation. Both the displacement field $u_i(X_i, t)$ and the strain field $\varepsilon_{ij}(X_i, t)$ are assumed in the analysis to be infinitesimal.

2.1.1. Viscoelastic fibre

For the simplification of the analysis, the continuum approach is maintained in this paper for the response of the single fibre. It is understood that the effect of fibre substructural parameters [8] will not be considered at the present stage of presentation. Hence, it is considered in the present analysis that the overall response of single fibre is of greater significance to the mechanical behaviour of a composite system.

From a phenomenological point of view, the concept of the relaxation function in linear viscoelasticity can be expressed [9] as follows

$$ \xi_{ij}(x_i, t) = E_{ijkl}(x_i, t)\varepsilon_{kl}(x_i, 0^+) + \int_{0+}^{t} J_{ijkl}(x_i, t-\tau)\varepsilon_{kl}(\tau)\,d\tau \tag{2}$$

where ξ_{ij} is the stress tensor, E_{ijkl} and J_{ijkl} are the tensorial elastic and relaxation functions, respectively. The solution of the relaxation kernel J_{ijkl} of Equation 2 has been considered for the uniaxial test situation by Haddad [10]. In view of the mathematical analysis carried out in the latter reference, Equation 2 may be written for the uniaxial case as

$$ \xi(t) = E\varepsilon + h(\varepsilon, b_1, b_2, \ldots) \times \sum_{I=1}^{N} D_I[\exp(F_I t) - 1] \tag{3}$$
$$ (I = 1, 2, \ldots, N)$$

in which the function $h(\cdot)$ accounts for the nonlinear hereditary effects and is given in a parametric form. In Equation 3, D_I, F_I (I = 1, 2, ..., N) and $b_1, b_2, \ldots$ are constants to be determined. The values of these constants can be established by a minimization procedure using available experimental data concerning the relaxation behaviour of the particular fibre material under consideration [10].

In the case of cellulosic fibres, for instance, the function $h(\cdot)$ in Equation 3 may be assumed [7] to take the simple form

$$ h(\cdot) = \exp(b\varepsilon) - 1 \tag{4}$$

where b is a material constant. Expanding asymptotically the exponential term in the form of $h(\cdot)$ given in Equation 4 and retaining only the first two terms, Equation 3 becomes

$$ \xi(t) = \left\{E - b\sum_{I=1}^{N} D_I[1 - \exp(F_I t)]\right\}\nabla u(t); \tag{5}$$
$$ (I = 1, 2, \ldots, N)$$

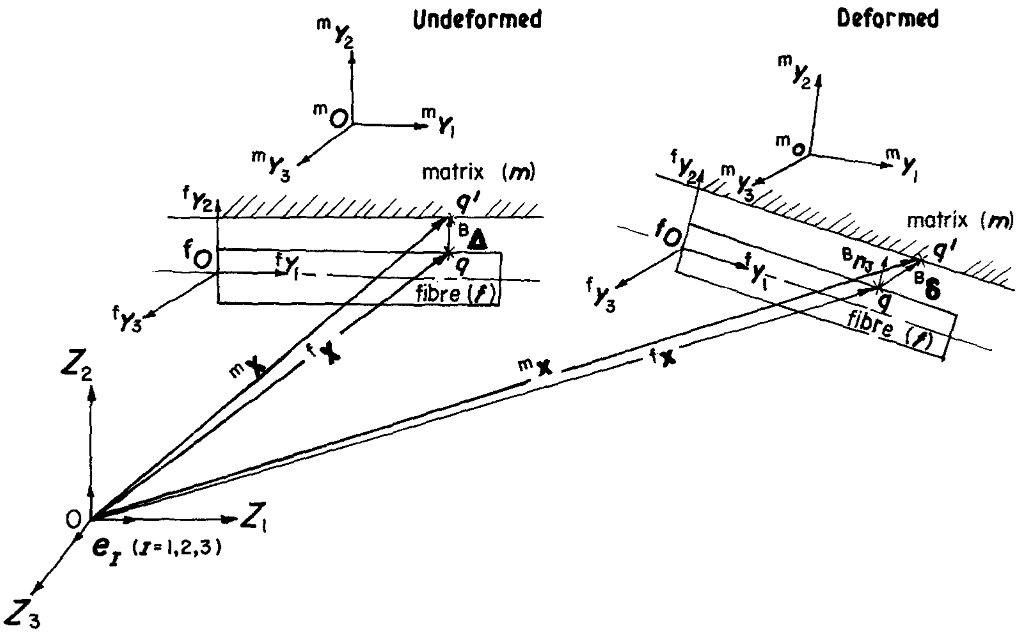

Figure 4 Deformation kinematics of fibre–matrix interface.

where ∇ is the gradient operator on the fibre microdeformation $u(t)$. Thus, if for instance from a system theoretical point of view, it is considered that the microstress in the fibre is the stimulus and the microdeformation is the corresponding response, then the operational equation for the response behaviour of viscoelastic fibre becomes:

$$^{f}\xi(t) = {}^{f}\Gamma(t)^{f}u(t) \qquad (6)$$

where $^{f}\Gamma(t)$ is a transform operator which takes, in view of Equation 5 the form:

$$^{f}\Gamma(t) = \left\{E - b\sum_{I=1}^{N} D_I[1 - \exp(F_I t)]\right\}{}^{f}\nabla \qquad (7)$$

$$(I = 1, 2, \ldots, N)$$

In generalized notations, however, Equation 6 may be written as:

$$^{f}\xi_{ij}(t) = {}^{f}\Gamma_{ijk}(t)^{f}u_k(t) \qquad (8)$$

This response behaviour relation will be used subsequently in the derivation of the response of the local domain (α).

2.1.2. Elastic matrix

The elastic response of the matrix material, at the local level, may be expressed in an operational form as

$$^{m}\xi_{ij}(t) = {}^{m}\Gamma_{ijk}{}^{m}u_k(t) \qquad (9)$$

where the material tensorial modulus $^{m}\Gamma_{ijk}$ is related to the generalized elastic tensor by

$$^{m}\Gamma_{ijk} = {}^{m}E_{ijks}{}^{m}\nabla_s \qquad (10)$$

in which $^{m}\nabla_s = \partial/\partial{}^{m}y_s$ is the gradient with respect to the matrix local coordinate frame.

2.1.3. Fibre–matrix interface

In any microstructural approach to the response of fibre composite systems, it is of utmost importance [11, 12] to include in the formulation the effect of the interfacial bonding between the fibres and the matrix. Studies on the subject are conventionally conducted in two ways, i.e. macroscopic and microscopic.

Within the macroscopic scheme, the approach is to estimate the shear modulus of the fibre–matrix interface from pull out tests on fibres that are embedded in the matrix to various lengths [e.g. 11, 13].

In the microscopic approach, however, significant research efforts were carried out [e.g. 14, 15], to explore the nature of adhesive bonding between different fibre materials and matrix substances particulars of composite structures. Due to the complexity of the molecular structure and of the surface conditions of the materials involved, there are still grave experimental difficulties which need to be overcome [8, 16]. One may, however, advance the argument that two types of bonding could be primarily responsible for the strength of the fibre–matrix interface, i.e. chemical, and the so-called physical (or frictional) bonding. The first is determined essentially by the compatibility of the molecular structure of the materials of the fibre and the matrix to form a particular type of chemical bonding. Such compatibility may be translated in terms of the type of matching atoms that might be available to form the bond, their separation vector, among other factors [17]. On the other hand, frictional bonding is due to the surface configuration of the two matching components and the extent of the contact forces that may develop between them during the heating and cooling cycles of the manufacturing process of the composite. It is also possible that the products of chemical reaction between certain types of fibre material and others of matrix substance accumulate on the surface of the fibre in large enough quantity and act as bonding between the fibre and the matrix. In this case, such an effect would be influenced by factors such as curing conditions and time, among others.

In the study of the deformation kinematics of the fibre–matrix interface (Fig. 4) the distance vector $^{B}\Delta$ between two matching points q and q' of the fibre and the matrix, respectively, is considered to be the basic kinematic parameter of an interfacial bond between the two components. The counterpart of this vector in the deformed state is denoted by ${}^{B}\boldsymbol{\delta}$ and the micro-deformation in the bond can thus be read as

$$ {}^{B}u_{I}(t) = {}^{B}\boldsymbol{\delta}_{I}(t) - {}^{B}\boldsymbol{\Delta}_{I} \quad (11) $$

where

$$ {}^{B}\boldsymbol{\Delta}_{I} = {}^{m}X_{I} - {}^{f}X_{I} \quad (12a) $$

and

$$ {}^{B}\boldsymbol{\delta}_{I}(t) = {}^{m}x_{I}(t) - {}^{f}x_{I}(t) \quad (12b) $$

Hence, by combining Equations 11 and 12, the interfacial deformation may be expressed, in view of Figs 3 and 4, by

$$ {}^{B}u_{I}(t) = {}^{m}u_{I}(t) - {}^{f}u_{I}(t) \quad (13) $$

In the case of chemical bonding, for instance, between matching the points of the fibre and the matrix, a "pair potential" form appears to be most suitable for the description of the bonding interaction. From classical considerations, one of the usual forms of such a potential in the one dimensional case is represented by "Morse function" [18] as follows:

$$ {}^{B}\Psi = {}^{B}\Psi_{0}[\exp(-2v^{B}u) - 2\exp(-v^{B}u)] \quad (14) $$

where ${}^{B}\Psi_{0}$ is the equilibrium value of the Morse potential at the value of ${}^{B}u = 0$ and v is the Morse constant. The material properties represented by the constants Ψ_{0} and v are obtainable from spectroscopic studies.

Based on the bonding potential form of Equation 14, an operational response behaviour relation for the bonding interaction may be expressed as [17]:

$$ {}^{B}\xi_{IJ}(t) = {}^{B}\Gamma_{IJK}{}^{B}u_{K}(t) \quad (15) $$

where the expression for the material operator ${}^{B}\Gamma_{IJK}$ can be written as

$$ {}^{B}\Gamma_{IJK} = \frac{-2\Psi_{0}v^{2}}{{}^{B}A(t)}\,{}^{B}n_{J}e_{K}e_{I}^{-1} \quad (16) $$

in which ${}^{B}A(t)$ is the actual bonded region within the fibre–matrix interface at time t. ${}^{B}n_{J}$ is the unit normal to the interface at the point of consideration and e_{K} is a unit base vector associated with the external co-ordinate frame (Fig. 4).

2.1.4. Response behaviour of a structural domain (α)

The load transfer to the local microstructure of the composite has been considered in the Appendix. In view of this analysis, the microstress in (α) may be expressed, in a probabilistic manner, by

$$ {}^{\alpha}\xi_{IJ}(t) = P_{1}{}^{f}\xi_{IJ}(t) + P_{2}{}^{m}\xi_{IJ}(t) \quad (17) $$

in which the probabilities P_{1} and P_{2} are expressed in terms of the microstructure (see Appendix, Equations A8 and A10). On the other hand, a relationship has been established between ${}^{m}\xi_{IJ}(t)$ and ${}^{f}\xi_{IJ}(t)$, namely

$$ {}^{m}\xi_{IJ}(t) = {}^{\alpha}\kappa(t){}^{f}\xi_{IJ}(t) \quad (18) $$

where ${}^{\alpha}\kappa(t)$ is a time-dependent function, characteristic of the microstructure, expressed by Equation A20. Hence, combining Equations 17 and 18, it follows that

$$ {}^{\alpha}\xi_{IJ}(t) = [P_{1} + P_{2}{}^{\alpha}\kappa(t)]{}^{f}\xi_{IJ}(t) \quad (19a) $$

or, in terms of the matrix microstress

$$ {}^{\alpha}\xi_{IJ}(t) = [P_{1}{}^{\alpha}\kappa^{-1}(t) + P_{2}]{}^{m}\xi_{IJ}(t) \quad (19b) $$

Substituting in Equation 19a the equivalence to the fibre microstress from Equation 8, the local response equation, valid within (α), becomes

$$ {}^{\alpha}\xi_{IJ}(t) = \{[P_{1} + P_{2}{}^{\alpha}\kappa_{IJ}(t)]{}^{f}\Lambda_{ijIJ}{}^{f}\Gamma_{ijk}(t)\}{}^{f}u_{k}(t) \quad (20) $$

where ${}^{f}\Lambda_{ijIJ}$ is a transformation of orientation matrix.

Introducing the transform operator ${}^{\alpha f}\gamma_{IJk}(t)$,

$$ {}^{\alpha f}\gamma_{IJk}(t) = \{[P_{1} + P_{2}{}^{\alpha}\kappa(t)]{}^{f}\Lambda_{ijIJ}{}^{f}\Gamma_{ijk}(t)\} \quad (21) $$

the response Equation 20 becomes

$$ {}^{\alpha}\xi_{IJ}(t) = {}^{\alpha f}\gamma_{IJk}(t){}^{f}u_{k}(t) \quad (22) $$

In a similar manner, by combining Equations 9 and 19b, one can write the structural domain response equation in terms of the matrix local deformation as

$$ \begin{aligned} {}^{\alpha}\xi_{IJ}(t) &= \{[P_{1}{}^{\alpha}\kappa^{-1}(t) + P_{2}]{}^{m}\Lambda_{ijIJ}{}^{m}\Gamma_{ijk}\}{}^{m}u_{k}(t) \\ &= {}^{\alpha m}\gamma_{IJk}(t){}^{m}u_{k}(t) \end{aligned} \quad (23) $$

where the material operator ${}^{\alpha m}\gamma_{IJk}(t)$ takes the form

$$ {}^{\alpha m}\gamma_{IJk}(t) = \{[P_{1}{}^{\alpha}\kappa^{-1}(t) + P_{2}]{}^{m}\Lambda_{ijIJ}{}^{m}\Gamma_{ijk}\} \quad (24) $$

Further, from Equations 22 and 23, a relationship can be expressed between the two material operators of (α) as follows:

$$ {}^{\alpha f}\gamma_{IJk}(t) = {}^{\alpha m}\gamma_{IJk}(t){}^{m}u_{k}(t){}^{f}u_{k}^{-1}(t) \quad (25) $$

2.2. Transition to the macroscopic response

Since the composite system that occupies a given physical domain is regarded in the present analysis as a discrete medium, a transition from the local description to the macroscopic one must be attempted. In this context, the concept of the intermediate domain, or mesodomain [6] is introduced. It is the smallest region of the medium on the boundary of which the macroscopic observables are still valid but on the other hand, is large enough to contain a statistical number of structural domains. This permits statistical principles to be introduced in the analysis. It is further postulated that within the macroregion of the system, the mesodomains are denumerable and non-intersecting such that:

$$ \bigcup_{M=1}^{\zeta}{}^{M}V = V \quad (26) $$

$$ {}^{M_1}V \cap {}^{M_2}V = \phi; \quad (M_1 \neq M_2) $$

where ζ designates the total number of mesodomains within a macrovolume V, ϕ is the null set and $\bigcup$, $\bigcap$ indicate the union and intersection of material domains, respectively.

In the case of a two dimensional composite system under loading in the Z_1-direction, for instance (Fig. 1), a mesodomain M ($M = 1, 2, \ldots, \zeta$) may be specified by the region bounded by the two theoretical scanning lines $S_1 - S_1$ and $S_2 - S_2$ which are perpendicular to the direction of external loading. The width of this domain is determined, following the above, in relation to the actual dimensions of the structural domain "α" such that, within "M", $\alpha = 1, 2, \ldots, {}^{M}N$. ${}^{M}N$ is very large. Within a mesodomain, all microscopic field quantities are considered to be stochastic functions of primitive random variables. Thus, the components of the fibre-microdeformation, for example, are seen as stochastic functions ${}^{f}u_K({}^{f}r_K, t)$. The latter can be regarded as a family of random variables ${}^{f}u_K({}^{f}r_K)_t$ depending on the time parameter t, or a family of curves ${}^{f}u_K(t)_{{}^{f}r}$ depending on the position vector ${}^{f}\boldsymbol{r}$.

At a particular time t, ${}^{f}u_K(t)$ can be expressed as a mean value over the mesodomain and a fluctuating component as

$$ {}^{f}u_K(t) = {}^{M}\langle {}^{f}u_K(t) \rangle + {}^{f*}u_K(t) \qquad (27) $$

where the fluctuating part ${}^{f*}u_K(t)$ is due to the random nature of the microstructure of the composite system and pertains locally to the particular fibre.

Letting ${}^{M}P(\cdot)$ denote the probability distribution of a random quantity within the mesodomain, then in view of the structural domain response (Equation 22), the probabilistic stress distribution within the mesodomain may be expressed by:

$$ {}^{M}P[{}^{\alpha}\xi_{IJ}(t)] = {}^{M}P[{}^{\alpha f}\gamma_{IJk}(t)]\,{}^{M}P[{}^{f}u_k(t)] \qquad (28a) $$

or, alternatively, with reference to Equation 23, as

$$ {}^{M}P[{}^{\alpha}\xi_{IJ}(t)] = {}^{M}P[{}^{\alpha m}\gamma_{IJk}(t)]\,{}^{M}P[{}^{m}u_k(t)] \qquad (28b) $$

In the present analysis, the mesoscopic quantities, valid on the boundary of the mesodomain, are taken as the expected values or the average of the corresponding microparameters. However, a more detailed analysis, as shown in [6], may take also higher moments of the relevant quantities in compliance with correlation theory [19]. Thus, with reference to Equation 25a, one may express the mesoscopic response relation by an averaging procedure over the mesodomain, i.e.,

$$ \begin{aligned} {}^{M}\sigma_{IJ}(t) &= {}^{M}\langle {}^{\alpha}\xi_{IJ}(t) \rangle = {}^{M}\langle {}^{\alpha f}\gamma_{IJk}(t) \rangle\, {}^{M}\langle {}^{f}u_k(t) \rangle \\ &= {}^{Mf}F_{IJk}(t)\,{}^{f}U_k(t) \end{aligned} \qquad (29) $$

where the mean value ${}^{M}\langle {}^{\alpha}\xi_{IJ}(t) \rangle$ may be considered, as a first approximation, to be associated with the value of the stress prescribed by the continuum mechanics theory on the boundary of the mesodomain. In this equation, the mesoscopic material operator ${}^{Mf}F_{IJk}(t)$ is expressed, Equation 21, by

$$ \begin{aligned} {}^{Mf}F_{IJk}(t) &= {}^{M}\langle {}^{\alpha f}\gamma_{IJk}(t) \rangle \\ &= \langle \{[P_1 + P_2{}^{\alpha}\kappa(t)]\,{}^{f}\Lambda_{ijIJ}\,{}^{f}\Gamma_{ijk}(t)\} \rangle \end{aligned} \qquad (30) $$

Also, with reference to Equation 28b, the mesoscopic response may be expressed in terms of the matrix deformation as

$$ \begin{aligned} {}^{M}\sigma_{IJ}(t) &= {}^{M}\langle {}^{\alpha}\xi_{IJ}(t) \rangle = {}^{M}\langle {}^{\alpha m}\gamma_{IJk}(t) \rangle\, {}^{M}\langle {}^{m}u_k(t) \rangle \\ &= {}^{Mm}F_{IJk}(t)\,{}^{m}U_K(t) \end{aligned} \qquad (31) $$

where, with reference to Equation 24, one can write

$$ {}^{Mm}F_{IJk}(t) = {}^{M}\langle \{[P_1{}^{\alpha}\kappa^{-1}(t) + P_2]\,{}^{m}\Lambda_{ijIJ}\,{}^{m}\Gamma_{ijk}\} \rangle \qquad (32) $$

It should be noted that the ensemble averages of the microstress and the microdeformation as shown above are taken with respect to their corresponding probabilistic distributions in the mesodomain. In this context, it should be mentioned that such distributions can be found experimentally for the microdeformation by use of stress-holographic interferometry technique [20].

The variance of the stress distribution (Equation 28) however, can be expressed by incorporating the fluctuating part of the random variable ${}^{\alpha}\xi_{IJ}(t)$. Hence, with reference to Equations 22 and 29, the variance becomes

$$ \begin{aligned} {}^{M}\langle {}^{\alpha *}\xi_{IJ}(t)\,{}^{\alpha *}\xi_{IJ}(t) \rangle = {}^{M}\langle [{}^{\alpha f}\gamma_{IJk}(t)\,{}^{f}u_K(t) \\ - {}^{Mf}F_{IJk}(t)\,{}^{f}U_k(t)]^2 \rangle \end{aligned} \qquad (33) $$

or, in terms of the microdeformations in the matrix, with reference to Equations 23 and 31, as

$$ \begin{aligned} {}^{M}\langle {}^{\alpha *}\xi_{IJ}(t)\,{}^{\alpha *}\xi_{IJ}(t) \rangle = {}^{M}\langle [{}^{\alpha m}\gamma_{IJk}(t)\,{}^{m}u_K(t) \\ - {}^{Mm}F_{IJk}(t)\,{}^{m}U_k(t)]^2 \rangle \end{aligned} \qquad (34) $$

It should, however, be mentioned that the mean value and the variance do not specify the stress distribution (Equation 28) uniquely. However, this may suffice for practical applications. This can be supported by the fact that the random distributions encountered in practice turn out to be "Gaussian". For a Gaussian distribution, the mean value and the variance completely specify the random variable.

3. Time-evolution of the internal deformation process

The internal deformation process corresponding to the mechanical response of the microstructure is considered, in this analysis, to be stochastic of the stationary Markov type [21]. Thus, the fibre deformation process $[{}^{f}u_K(t); t > 0]$, the matrix deformation process $[{}^{m}u_K(t); t > 0]$ and the interfacial binding deformation $[{}^{B}u_K(t); t > 0]$ are regarded to be independent, time-wise continuous processes of the Markovian character. Accordingly, the probability distribution of any of the above deformations $u_K(t)$ may be considered to be completely defined for all $t > \tau$ by the value assumed at $t = \tau$ and, in particular, is independent of the history of such process for all $0 < t < \tau$.

Analytically, a Markov process is completely determined by its transition probabilities. Letting $\mathbf{p}^{u}(t)$ denote the transition probabilities of the stochastic process $[u_K(t); t > 0]$, then the time history of the probability distribution $P[u_K(t)]$ can be described by

$$ P[u_K(t)] = \mathbf{p}^{u}(t - \tau)P[u_K(\tau)] \qquad (35) $$

Under the restriction that the process $[u_K(t); t > 0]$ is represented by a stationary Markov process, $p^{u}(t)$ is governed by the Chapman–Kolmogorov [21] equation

$$ \mathbf{p}^{u}(t) = \mathbf{p}^{u}(t)p^{u}(0) \qquad (36) $$

and the backward and forward Kolmogorov equations given, respectively, by

$$ \begin{aligned} \frac{d\mathbf{p}^{u}(t)}{dt} &= \mathbf{p}^{u}(t)\mathbf{Q}^{u}(t) \\ \frac{d\mathbf{p}^{u}(t)}{dt} &= \mathbf{Q}(t)\mathbf{p}^{u}(t) \end{aligned} \qquad (37) $$

This is subject to the initial conditions: $\mathbf{p}^{u}(0) = \mathbf{I}$, ($\mathbf{I}$ is the identity matrix). In Equation 37, $\mathbf{Q}(t)$ is a time-dependent transition probability matrix, given by

$$\mathbf{Q}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{p}^{u}(t, t + \Delta t)}{\Delta t} \tag{38}$$

However, for simplification of the analysis, one may assume that the transition probability matrix is time independent, i.e.,

$$\mathbf{Q}(t) = \mathbf{Q}$$

This suggests that the solution of the Kolmogorov equations given in Equation 37 can be written as [21]

$$\mathbf{p}^{u}(t) = \exp(\mathbf{Q} \cdot t) \tag{39}$$

Thus, by combining Equations 35 and 39, the former becomes

$$P[u_K(t)] = \exp[\mathbf{Q} \cdot \Delta t] P[u_K(\tau)] \tag{40}$$

where $\Delta t = t - \tau$.

Hence, the probabilistic distribution of microdeformation within the mesodomain at any time $t > \tau$ can be found from the corresponding distribution at $t = \tau$. On the other hand, if two successive distributions of deformation could be assessed experimentally, a value for the transition probability matrix $\mathbf{Q}$ would be obtained. An important objective of the present research is to determine experimentally the matrix $\mathbf{Q}$ using high magnification scanning techniques.

In Fig. 5, the time evolution of the process $[u_K(t); t > 0]$ is demonstrated in connection with its maximum limit, i.e. $|u_k|_{max}$. In this figure, it is of interest to note that a failure criterion of the component, within the mesodomain, may be conjectured by setting:

$$H(t) = 1 - \int_{|u_K|_{max}}^{\infty} d^{M}P[u_K(t)] \tag{41}$$

where $u_{K\,max}$ is a characteristic of the particular composite material under consideration.

4. Conclusion

The present work has been mainly aimed at formulating a deformation theory concerning two-dimensional composite systems with randomly oriented, short viscoelastic fibres. For this purpose, the relevant field quantities characterizing either the geometrical or physical properties of the microstructure have been considered from the onset as random variables or functions of such variables. The specific features of the approach are as follows.

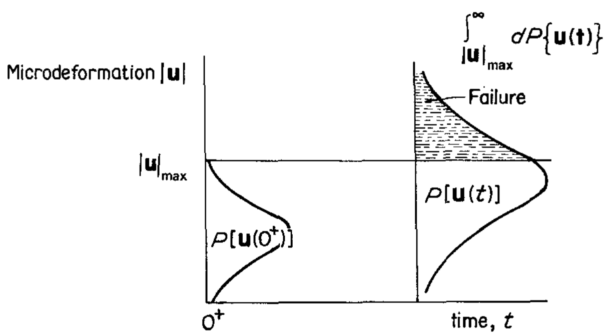

Figure 5 Time-evaluation of the deformation process.

(i) The inclusion of the most significant material characteristics of a composite structure into the formulation. This has been made possible by the application of a "material operator" which is a function of actual microstructural properties of a given composite site.

(ii) The presentation of possible internal distributions of stresses. The knowledge of such stress distributions is of considerable importance in engineering practice.

(iii) Establishing the time-evolution of the internal deformation process leading to the final failure of the composite system.

The theoretical analysis has been developed in a generalized manner that may be applied to a large class of fibrous systems.

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Appendix I: Load transfer to the microstructure

In Fig. 1, a mesodomain of the composite specimen is identified by the region between two scanning lines S_1–S_1 and S_2–S_2. The latter, as discussed earlier, are perpendicular to the direction of the external, uniaxial loading $b_I : (b_1, 0, 0)$. Accordingly, b_1 is transmitted equivalently to the specimen cross-sectional areas at S_1–S_1 and S_2–S_2. The corresponding (continuum) stress may be defined (Fig. A1) as

$${}^{M}\sigma_{IJ} = \frac{b_I {}^{M}n_J}{\chi C} = \frac{b_I {}^{M}n_J}{\chi(Wh)} \tag{A1}$$

where ${}^{M}\boldsymbol{n}_J$ ($J = 1, 2, 3$) is the unit normal vector to C, C is the cross-sectional area of the specimen perpendicular to the lead direction, χ is the matrix void ratio (assumed to be constant throughout the specimen), W designates the specimen width and h is the specimen thickness.

With reference to Fig. A1, the scanning line S_1–S_1, for instance, may intersect a fibre f, that is inclined to the Z_1-direction under an angle ${}^{f}\theta$, over a certain length ${}^{f}q$. At the same time, the specimen cross-section at S_1–S_1 will intersect the same fibre through an area ${}^{f}s$. At any particular instant of time, the forces exerted on ${}^{f}s$ are statistically equivalent to a force ${}^{f}b_I(t)$ and a couple of moments ${}^{f}M_I(t)$. However, for simplification of the analysis and due to the fact that experimental values are often obtainable from uniaxial test situations, the force vector ${}^{f}\boldsymbol{b}(t)$ is shown in Fig. A1. In case of uniaxial external loading in the Z_1-direction, ${}^{f}\boldsymbol{b}(t)$ will have the components (${}^{f}\boldsymbol{b}_1(t)$, 0, 0) and its magnitude may be approximated by the relation

$${}^{f}\boldsymbol{b}_1(t) = \frac{{}^{f}s}{\chi C} b_1 = \left[\frac{\pi({}^{f}r)^2}{\chi C(\cos^{f}(\theta)}\right] b_1 \tag{A2}$$

in which ${}^{f}r$ is the radius of the fibre, assumed to be constant throughout the specimen.

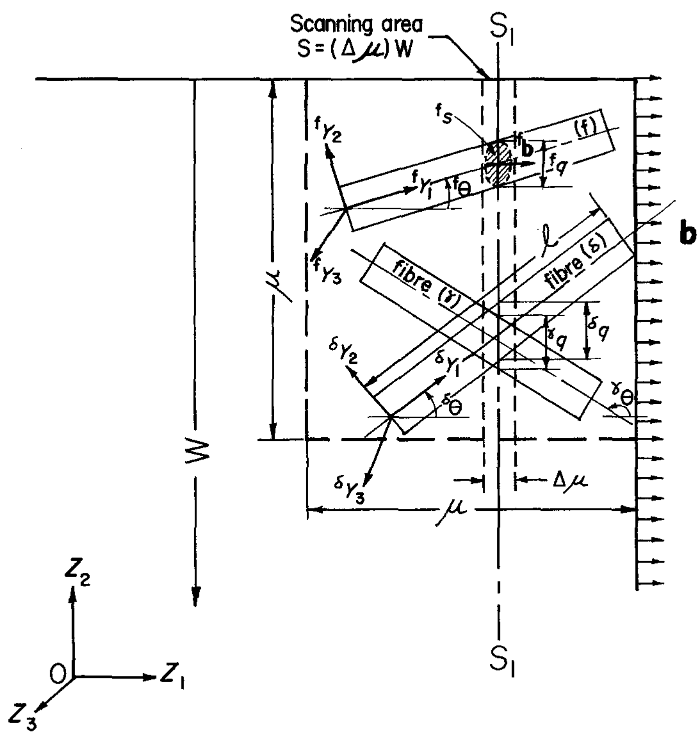

Figure A1 Load transfer to the microstructure.

On the other hand, the scanning line S_1–S_1 may intersect a crossing of two fibres, say δ and γ. In this case, the fibre load contributing to the external loading may be expressed, following Equation A.2, by

$$ {}^{j}b_1 = \frac{1}{2}\left(\frac{{}^{\delta}s + {}^{\gamma}s}{\chi C}\right) b_1 = \frac{\pi({}^{f}r)^2}{2C} \times \left[\frac{\cos{}^{\delta}\theta + \cos{}^{\gamma}\theta}{\cos{}^{\delta}\theta \cos{}^{\gamma}\theta}\right] b_1 \qquad \text{(A3)} $$

with the understanding that ${}^{j}b_1 \simeq {}^{\delta}b_1 \simeq {}^{\gamma}b_1$ in this particular situation. Also, there exists the probability that the scanning line S_1–S_1 may intersect, at the same point in question, a matrix material between two neighbouring or crossing fibres. The load contributed by the matrix material, within the local domain α, to the external loading is denoted by ${}^{m}b_I : ({}^{m}b_1, 0, 0)$. Hence, in view of the earlier definition of α, the load ${}^{m}b_1$ may be expressed as

$$ {}^{m}b_1(t) = \frac{{}^{m}s}{C} b_1(t) = \left[\frac{\left({}^{\alpha}s - \sum_{f=1}^{\alpha_n} {}^{f}s\right)}{C}\right] b_1(t) $$

$$ = \left[\frac{\left({}^{\alpha}s - \sum_{f=1}^{\alpha_n} \frac{\pi({}^{f}r)^2}{\cos{}^{f}\theta}\right)}{C}\right] b_1(t) \qquad \text{(A4)} $$

where ${}^{\alpha}s = {}^{m}s + \Sigma_{f=1}^{\alpha n}\, {}^{f}s$, the cross-sectional area of the local domain, α, as intersected by S_1–S_1 and ${}^{\alpha}n$ is the probable number of fibres intersecting with the scanning line S_1–S_1 within α.

In view of the above, one may introduce the following probabilities

${}^{f}p$, the probabilities that the scanning line S_1–S_1 will intersect, at a certain point of the specimen, a single fibre,

${}^{j}p$, the probabilities that S_1–S_1 may intersect, at the same point, a crossing between two intersecting fibres, and

${}^{m}p$, the probabilities that S_1–S_1 will intersect, at the same point in question, a matrix material only.

With the understanding that $0 \leqslant {}^{f}p, {}^{j}p, {}^{m}p \leqslant 1$,

$$ {}^{f}p + {}^{j}p + {}^{m}p = 1. \qquad \text{(A5)} $$

In terms of these probabilities, the probabilistic value of the microstress, at the point under consideration, may be expressed as

$$ {}^{\alpha}\xi_{IJ}(t) = {}^{f}p\,{}^{f}\xi_{IJ}(t) + {}^{j}p\,{}^{j}\xi_{IJ}(t) + {}^{m}p\,{}^{m}\xi_{IJ}(t) \qquad \text{(A6)} $$

where ${}^{j}\xi_{IJ}(t)$ is the microstress at the crossing between two fibres. The value of the latter may be approximated, for the present case of uniaxial loading and following Equation A3, by

$$ {}^{j}\xi_{IJ}(t) = \tfrac{1}{2}\,{}^{f}\xi_{IJ}(t) \qquad \text{(A7)} $$

Combining Equations A5, A6 and A7, one may express Equation A6 in the following form (as Equation 17):

$$ {}^{\alpha}\xi_{IJ}(t) = P_1\,{}^{f}\xi_{IJ}(t) + P_2\,{}^{m}\xi_{IJ}(t) \qquad \text{(A8)} $$

in which $P_1 = {}^{f}p + \frac{1}{2}{}^{j}p$ and $P_2 = 1 - {}^{f}p - {}^{j}p$.

It is now required to determine the two probabilities ${}^{f}p$ and ${}^{f}p$ so that Equation A8 can be made.

Denoting by N the total number of fibres that may intersect with S_1–S_1 and by G the probable number of crossings that may occur between the N fibres along S_1–S_1, then, one may write

$$ {}^{f}p = \frac{\sum_{f=1}^{N-2G} (2\,{}^{f}r/\cos{}^{f}\theta)}{W} \qquad \text{(A9a)} $$

and

$$ {}^{j}p = \frac{\sum_{f=1}^{G} (2\,{}^{f}r/\cos{}^{f}\theta)}{W} \qquad \text{(A9b)} $$

Hence, by combining Equations A8 and A9, it follows that:

$$P_1 = \frac{1}{W}\left[\sum_{f=1}^{N-2G} (2^f r/\cos^f\theta) + \frac{1}{2}\sum_{f=1}^{G} (2^f r/\cos^f\theta)\right] \quad \text{(A10a)}$$

and

$$P_2 = 1 - \frac{1}{W}\left[\sum_{f=1}^{N-G} (2^f r/\cos^f\theta)\right] \quad \text{(A10b)}$$

In the above equation, the number of fibres N is a function of basic geometric and physical characteristics of the fibrous material within the composite. It is often expressed in relation to the two principal axes of the orientation pattern of the fibres within the composite sheet. Hence, following [22], one may write that

$$N = N_{MD} = \frac{\Xi W}{\Omega}\int_0^{\pi} \sin(^f\theta)p(^f\theta)\, d^f\theta \quad \text{(A11a)}$$

(when S_1–S_1 is perpendicular to the major axis, machine direction, of the fibre orientation pattern within the composite) or

$$N = N_{CD} = \frac{\Xi W}{\Omega}\int_{-\pi/2}^{\pi/2} \cos^f\theta p(^f\theta)\, d^f\theta \quad \text{(A11b)}$$

(when S_1–S_1 is perpendicular to the cross-directional axis of the orientation pattern of the composite), in which Ξ is the basic weight [22] of the fibrous material within the composite, Ω is the weight per unit length of the fibre and $p(^f\theta)$ is the fibre orientation distribution function.

On the other hand, the number of junctions G intersecting with S_1–S_1 is determined essentially by the knowledge of the probable number of fibres N, expressed above, and the random orientation of their axes over the plane of the sheet. For this purpose, we introduce the scanning area S whose major axis is the scanning line S_1–S_1, as shown in Fig. A1. The width $\Delta\mu$ of this area is infinitesimal, i.e. $\Delta\mu \ll 2^f r$, such that a crossing occurring between two fibres in S must intersect with S_1–S_1. Here, we consider a fibre (δ) of length l lying in the plane of the scanning area, with its axis making an angle ($^\delta\theta$) with the direction of the Z_1-axis, as shown in Fig. A1. We also consider another fibre (γ), of same length l, dropped into the scanning area at random with an angle $0 \leqslant {^\gamma\theta} \leqslant 2\pi$ between its axis and the Z_1-direction. The probability of interception between the two fibres in the scanning area may therefore be written as [23] $(l^2/S)\sin(^\delta\theta - {^\gamma\theta})$. Hence, by employing the probability density function of the angle $^f\theta$, i.e. $p(\theta)$, and considering that all fibres are of the same length, the probability of interception between two fibres in S may be expressed, on the average, as:

$$\eta = \frac{l^2}{S}\int_0^{\pi}\int_0^{\pi} \sin(^\delta\theta - {^\gamma\theta})p(^\delta\theta)\, d^\delta\theta\, p(^\gamma\theta)\, d^\gamma\theta \quad \text{(A12)}$$

Letting L represent the number of ways in which pairs of fibres can be selected within the scanning area, then the probable number of interceptions between the fibres in S may be written as

$$G = \eta L \quad \text{(A13)}$$

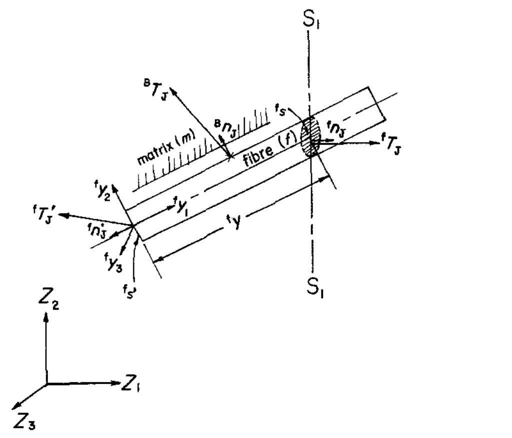

Figure A2 Equilibrium of a fibre embedded in matrix material.

However, in order to calculate L in (A13), one may consider one of the N fibres in S: it can intercept all the fibres in S except itself or make $(N - 1)$ interceptions. Considering all of the N fibres in S gives $N(N - 1)$ combinations of two fibres. As each fibre is counted twice, then

$$L = \tfrac{1}{2}N(N-1) = \tfrac{1}{2}N^2 : N \gg 1 \quad \text{(A14)}$$

Combining Equations A13 and A14, the probable number of crossings between fibres along S_1–S_1 is given by

$$G = \frac{\eta N^2}{2} \quad \text{(A15)}$$

in which η is given by Equation A12. In the case of non-oriented fibrous pattern within the composite, i.e. when the fibre angular distribution is circular, the function $p(\theta)$ appearing in Equations A11 and A12 may be expressed [23] by $p(\theta) = 1/\pi$ for a circular, angular distribution, and hence it can be shown, in this particular case, that

$$G = \frac{N\Xi l^2}{\pi^2\mu\Omega} \quad \text{(A16)}$$

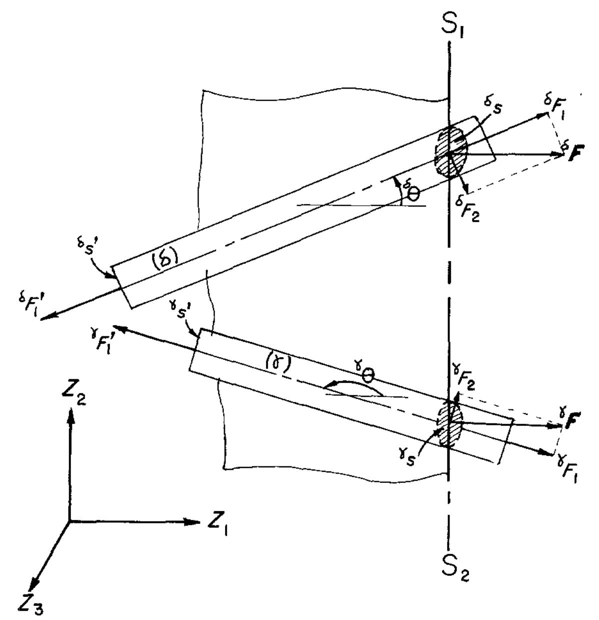

Figure A3 Equilibrium of the connection between two fibres δ and γ.

one dimensional case that

$$ {}^{f}\lambda = \frac{\sin{}^{\delta}\theta[\cos^{2}{}^{\gamma}\theta + \cos^{2}{}^{\delta}\theta - \cot{}^{\delta}\theta(\sin{}^{\gamma}\theta\cos{}^{\gamma}\theta + \sin{}^{\delta}\theta\cos{}^{\delta}\theta)]}{\cos{}^{\delta}\theta\sin({}^{\gamma}\theta - {}^{\delta}\theta)} \quad \text{(A21)} $$

Further, it is necessary to establish the interdependence between the local stresses, i.e. ${}^{f}\xi_{IJ}(t)$, ${}^{B}\xi_{IJ}(t)$ and ${}^{m}\xi_{IJ}(t)$ for a particular fibre f embedded in the matrix m. Thus, with reference to Fig. A2, the unit normal vector to the fibre section area ${}^{f}s$, at S_1–S_1, is denoted by ${}^{f}\boldsymbol{n}_J$ and the associated traction vector is designated by ${}^{f}T_J$. The corresponding vectors acting on another sectional area ${}^{f}s'$, of the same fibre are, respectively, ${}^{f}\boldsymbol{n}'_J$ and ${}^{f}\boldsymbol{T}'_J$. In the analysis, the resultant effect of the two tractions are taken to be equivalent to the effect of the interfacial traction vector ${}^{B}\boldsymbol{T}_J$ acting between the fibre and the matrix. Hence, one may write that

$$ {}^{f}\boldsymbol{T}_J(t){}^{f}s - {}^{f}\boldsymbol{T}'_J(t){}^{f}s' = {}^{B}\boldsymbol{T}_J(t){}^{f}c{}^{f}y \quad \text{(A17)} $$

where ${}^{f}c$ is the circumference of the fibre in contact with the matrix and ${}^{f}y$ is a variable indicating the length along the fibre between the sectional areas ${}^{f}s$ and ${}^{f}s'$.

In terms of the microstresses corresponding to the traction vectors in (A17), the latter becomes

$$ [{}^{f}\xi_{IJ}(t){}^{f}n_J]{}^{f}s - [{}^{f}\xi_{IJ}(t){}^{f}n'_J]{}^{f}s = [{}^{B}\xi_{IJ}(t){}^{B}n_J]{}^{f}c{}^{f}y \quad \text{(A18a)} $$

or

$$ {}^{B}\xi_{IJ}(t) = \frac{{}^{B}n_J^{-1}}{{}^{f}c{}^{f}y}[{}^{f}n_J{}^{f}s - {}^{f}n'_J{}^{f}s'{}^{f}\lambda(t)]{}^{f}\xi_{IJ}(t) \quad \text{(A18b)} $$

where

$$ {}^{f}\lambda(t) = {}^{f}\xi'_{IJ}(t)/{}^{f}\xi_{IJ}(t) \quad \text{(A18c)} $$

Further, by combining Equations 9, 13, 15 and A18a, it is possible to write a relation between the matrix microstress ${}^{m}\xi_{IJ}(t)$ and that of the fibre ${}^{f}\xi_{IJ}(t)$ as (Equation 18)

$$ {}^{m}\xi_{IJ}(t) = {}^{\alpha}\kappa(t){}^{f}\xi_{IJ}(t) \quad \text{(A19)} $$

in which ${}^{\alpha}\kappa(t)$ takes the form

$$ {}^{\alpha}\kappa(t) = {}^{m}\Gamma_{IJS}\left[{}^{B}\Gamma^{-1}_{IJS}\frac{{}^{B}n_J^{-1}}{{}^{f}c{}^{f}y}({}^{f}n_J{}^{f}s - \lambda{}^{f}n'_J{}^{f}s') + {}^{f}\Gamma^{-1}_{IJK}\delta_{SK}\right] \quad \text{(A20)} $$

where δ_{SK} is the Kronecker delta.

It is now required to determine the parameter ${}^{f}\lambda(t)$ of Equation A18b so that Equations A18a and A19 can be made. In this regard, one may consider for the simple one-dimensional case, the equilibrium of the connection between two fibres δ and γ, for instance, under the effect of internal axial forces ${}^{\delta}F$, ${}^{\delta}F'_1$, ${}^{\gamma}F_1$ and ${}^{\gamma}F'_1$ acting at the ends of the fibres as shown in Fig. A3. In this figure, the orientation of the two fibres is given respectively by ${}^{\delta}\theta$ and ${}^{\gamma}\theta$. Hence, following the analysis carried out by Haddad [24], it can be shown for the

It is assumed in the present analysis that the fibre orientation angle is experimentally determinable [7], hence the factor ${}^{f}\lambda$ can be determined by employing Equation A21.

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